Answers

1. The denominator is zero at \( x = 2 \) and \( x = -2 \). When \( x = 2 \), the numerator is also zero, so a limit may exist, but when \( x = -2 \), the numerator is nonzero so there can be no limit. Checking the limit at \( x = 2 \),
\[
\lim_{x \to 2} \frac{x - 2}{(x - 2)(x + 2)} = \lim_{x \to 2} \frac{1}{x + 2} = \frac{1}{4}
\]
Since this limit exists, but the function’s value is not defined at \( x = 2 \), there is a removable discontinuity there (and only there). The continuous extension is the same function, but with the value defined to equal the limit at \( x = 2 \):
\[
g(x) = \begin{cases} 
\frac{x - 2}{x^2 - 4} & x \neq 2 \\
\frac{1}{4} & x = 2 
\end{cases}
\]

2. The denominator is zero at \( x = 0 \), and so is the numerator, so there may be a limit at that point. We can use L’Hôpital’s rule (twice) to check:
\[
\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \lim_{x \to 0} \frac{\sin x}{2x} = \lim_{x \to 0} \frac{\cos x}{2} = \frac{1}{2}
\]
Since this limit exists, but the function’s value is not defined at \( x = 0 \), there is a removable discontinuity there (and only there). The continuous extension is the same function, but with the value defined to equal the limit at \( x = 0 \):
\[
g(x) = \begin{cases} 
\frac{1 - \cos x}{x^2} & x \neq 0 \\
\frac{1}{2} & x = 0 
\end{cases}
\]

3. The denominator is zero at \( x = 0 \), and so is the numerator, so there may be a limit at that point. We can use L’Hôpital’s rule to check:
\[
\lim_{x \to 0} \frac{\tan x}{x} = \lim_{x \to 0} \frac{\sec^2 x}{1} = \frac{1}{1} = 1
\]
Since this limit exists, but the function’s value is not defined at \( x = 0 \), there is a removable discontinuity there (and only there, since that’s the only place the denominator is zero). The continuous extension is the same function, but with the value defined to equal the limit at \( x = 0 \):
\[
g(x) = \begin{cases} 
\tan x & x \neq 0 \\
1 & x = 0 
\end{cases}
\]

4. TRUE: a differentiable function is continuous, and continuity is the requirement for the intermediate value theorem to apply.

5. FALSE: The theorem does state that such a point \( c \) must exist, but it says nothing about how to find it. A counterexample is two continuous functions, both having the given values at \( x = 0 \) and \( x = 10 \), but passing through \( y = 2 \) at different points.

6. TRUE: if the value at two ends of an interval is the same, the theorem does not tell you anything about the values of the function in between.
7. FALSE: The theorem states that there is at least one such point $c$, but there could be more than one. A counter-example is any continuous function that has the given values at $x = 0$ and $x = 10$, but oscillates around $y = 50$ in between.

8. FALSE: The function is not continuous on the interval $[-1, 1]$, so the Intermediate Value Theorem does not apply in this case, and in fact, there is no $c$ for which $\frac{1}{c} = 0$.

9. The cosine function and the function $g(x) = x$ are both continuous, so their difference is continuous for all real numbers (hence continuous on the interval $[0, 1]$). Evaluating at $x = 0$, we get $f(0) = 1$, and evaluating at $x = 1$, we get $f(1) = \cos(1) - 1$. We know cosine is 1 at $x = 0$ and any multiple of $2\pi$, but is less than 1 everywhere else, and so $\cos(1) < 1$, which means $\cos(1) - 1 < 0$. Now, because the function’s value is greater than 0 at $x = 0$, and less than 0 at $x = 1$, it must (by the Intermediate Value Theorem) have a value of 0 for at least one point $c$ in $[0, 1]$.

10. The sine function is continuous everywhere, and so is $\frac{x+1}{3}$, so the square of the sine function is continuous, and so is the sum of two continuous functions, so $f(x)$ is continuous everywhere. Evaluating at the end points of each interval:

$f(-2) = \sin^2(-\pi) - \frac{-1}{3} = \frac{1}{3}$

$f(-1) = \sin^2(-\frac{\pi}{2}) - \frac{0}{3} = 1$

$f(0) = \sin^2(0) - \frac{1}{3} = -\frac{1}{3}$

$f(1) = \sin^2(\frac{\pi}{2}) - \frac{2}{3} = \frac{1}{3}$

$f(2) = \sin^2(\pi) - 1 = -1$

(a) On the interval $[-2, -1]$, the function is positive at both ends, so we CANNOT state that it passes through zero in that interval (that does not automatically mean that it does not - just that we can’t state for certain that it does).

(b) On the interval $[-1, 0]$, the function is positive at the left end and negative at the right end, so by the IVT, it must have the value 0 at some point in the interval.

(c) On the interval $[0, 1]$, the function is negative at the left end and positive at the right end, so by the IVT, it must have the value 0 at some point in the interval.

(d) On the interval $[1, 2]$, the function is positive at the left end and negative at the right end, so by the IVT, it must have the value 0 at some point in the interval.